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ELECTRODYNAMICS AND WAVE PROPAGATION

A Simplified Method for the Analysis of Eigenmodes in Random and Ordered Media

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Abstract—A simplified method for modeling eigenmodes in inhomogeneous media is proposed. The method is based on the numerical solution of the steady-state wave equation with a specified distribution of dielectric permittivity. This method is used for the analysis of both random and ordered media of different dimensions. It is shown that localized resonant modes arise only in the absence of complete ordering and effective resonators forming such modes comprise interstices between particles and adjoining particles. It is found that resonant modes in random media exist only within a fixed interval of values of ratio b/λ, where b is the particle size and λ is the wavelength ($b/\lambda < 1$). Localized modes arise at larger values of this ratio as compared to delocalized modes.

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INTRODUCTION

Since the publication of the first paper predicting the possibility of lasing in a randomly inhomogeneous medium [1] and the first experimental observation of lasing in powders [2], a large number of experimental and theoretical studies of this effect have been reported. However, much still remains unclear, in particular, the spatial distribution of regions generating laser radiation in randomly inhomogeneous media. It has been found experimentally that, in such media, lasing arises in small localized parts of a medium rather than throughout the excited region. This situation occurs in both powders consisting of micron particles [3] and zinc oxide powders with a particle size of only several nanometers [4]. This fact points to the presence of localized resonant modes in these media, a phenomenon that looks strange at first sight because these media do not contain any resonant elements.

Existence of resonant modes in a 2D randomly inhomogeneous medium has been demonstrated clearly in [5, 6] by means of numerical solution of the Maxwell equations with the finite-difference time domain (FDTD) method. This approach can be used to both obtain the spatial distribution of the mode field and estimate the mode quality factor. However, the FDTD method is very laborious. For this reason, analysis of a 3D medium or consideration of a large number of realizations of a random medium and a set of modes for each realization is very difficult. At the same time, such analysis is of great interest for understanding features of the light generation process and the localization effect in random media. Therefore, we propose a simplified method for modeling eigenmodes of the electromagnetic field in such media. It follows from the Maxwell equations that eigenfrequencies and amplitude distributions of the mode fields are determined by the steady-state wave equation. Therefore, the method proposed is based on the numerical solution of the steady-state wave equation with a specified spatial distribution of dielectric permittivity.

1. THE APPROACH

Let us first consider a 2D medium. It can be shown readily that the amplitude of steady-state TE oscillations in such medium must satisfy the 2D wave equation (see, e.g., [7]). In order to find the numerical solution to this equation, it should be discretized, i.e., rewritten in terms of differences between amplitudes of field **E** at adjacent points of the considered medium. This procedure results in the following system of equations:

$$E_{i+1,j} + E_{i,j+1} + E_{i-1,j} + E_{i,j-1} - \left(4 - \frac{\omega^2}{c^2} b^2 \varepsilon_{i,j}\right) E_{i,j} = 0, \quad (1)$$
$$i = 1 \ldots m, \quad j = 1 \ldots n.$$

Here, i and j are the numbers of the discretization nodes situated along axes x and y, respectively; b is the spacing between adjacent nodes; ω is the angular frequency of electromagnetic oscillations; c is the speed of light; and $\varepsilon_{i,j}$ is the dielectric permittivity of the medium at node (i, j). Thus, discretization reduces this differential equation to a system of homogeneous linear equations. The number of equations coincides with total number of discretization nodes mn, where m and n are the numbers of nodes along axes x and y, respectively. Solving system (1), we should specify values of field **E** at the

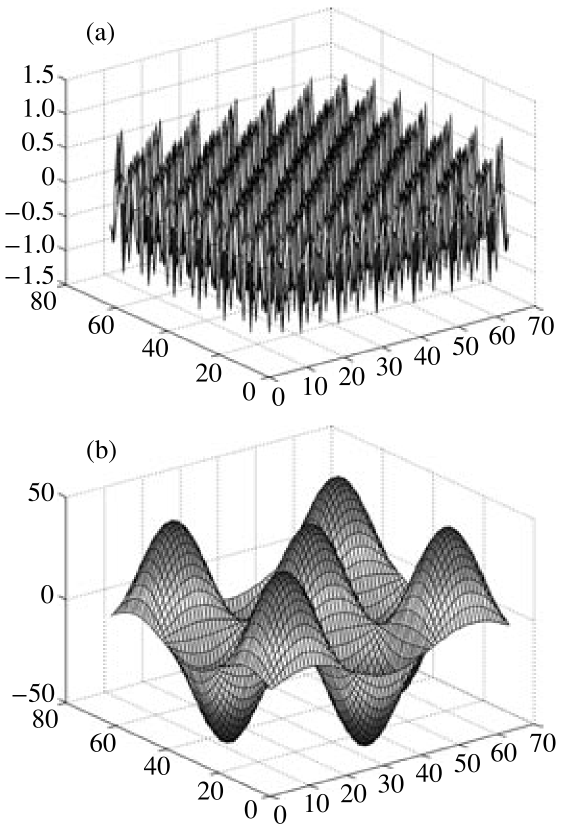

Fig. 1. Modes in an ordered media for (a) α = 0.5919483140805991 ($b/\lambda \approx 0.122$) and (b) α = 0.011960417020662 ($b/\lambda \approx 0.017$).

points $i - 1 = 0, j - 1 = 0$ and $i + 1 = m + 1, j + 1 = n + 1$, i.e., boundary conditions. Setting $E_{0,j} = sE_{1,j}$, $E_{i,0} = sE_{i,1}$ and $E_{m+1,j} = sE_{m,j}$, $E_{i,n+1} = sE_{i,n}$, we can consider different boundary conditions by choosing different values of factor s. Here, we consider the simplest variant, $s = 0$, because preliminary calculations have shown that introduction of $s \neq 0$ does not cause fundamental changes in the results.

In order to solve system (1), i.e., to find the eigenfrequency and the spatial distribution of the amplitude of the resonant mode, we divide each equation of the system by $\varepsilon_{i,j}$. As a result, we arrive at an eigenvalue search problem for the eigenvalues:

$$\alpha = \frac{\omega^2 b^2}{c^2} = \left(\frac{2\pi b}{\lambda}\right)^2.$$

For several eigenvalues, we determined their eigenvectors and, as a result, the distribution of the field amplitude (the shape of the resonant mode). All calculations were performed with MATLAB.

The 2D medium was modeled with the help of a matrix whose elements take on two different values. One of these values corresponds to dielectric permittivity ε of the medium particles, while the other value corresponds to dielectric permittivity ε_1 of interstices between the particles (usually, $\varepsilon_1 = 1$). In the simplest approximation used in this paper, elements of this matrix are, respectively, particles of the material and interstices between the particles. Hence, in this model, the particle size is approximately equal to spacing b between nodes. Since we model zinc oxide here, $\varepsilon = 6$ ($6 \approx n^2$, where $n = 2.45$ is the refractive index of excited zinc oxide [8]). The ratio of the number of elements with $\varepsilon = 6$ (the number of particles) to the total number of the matrix elements determines filling factor of the medium, Φ. In initial calculations, we used $\Phi = 50\%$, i.e., $\varepsilon = 6$ was assigned to half the elements and $\varepsilon = 1$ to the other half. Additional calculations were performed with different values of the filling factor.

Considering different variants of the 2D medium, we tried to answer the following questions: (i) What are the shapes of the field distributions of resonant modes? (ii) Are these modes localized or diffuse, chaotic or ordered? (iii) What is the interval of eigenvalues (values of ratio b/λ) in which a particular mode type is realized? (iv) What roles do the filling factor and the dielectric permittivity of the material particles serve? In order to answer these questions, we performed calculations in a wide range of eigenvalue α (i.e., b/λ) for both an ordered 2D medium and different realizations of a random medium. The results of these calculations are represented in the form of figures in which the vertical coordinate is proportional to the field amplitude at the point with horizontal coordinates (i, j).

2. ORDERED TWO-DIMENSIONAL MEDIUM

Let us begin with the analysis of an ordered matrix-medium in which particles with $\varepsilon = 6$ are arranged in staggered order. This is the model of the simplest 2D photonic crystal. Resonant modes in this medium are not localized but distributed in a certain order throughout the medium (Fig. 1a). As ratio b/λ decreases, the field distributions of the modes existing in the ordered medium tend toward the distributions typical of standing waves (Fig. 1b).

The minimum eigenvalue is α = 0.001334200099938 ($b/\lambda \approx 0.0058$). The distribution of the corresponding "minimum" mode is shown in Fig. 2. It may be of interest that minimum modes existing in media of any dimension, both ordered and random, have a similar distribution. Apparently, this means that, at $\lambda \gg b$, the field distribution corresponds to that in a continuous medium.

The sequence of eigenvalues ends with α = 0.6661996228385793 ($b/\lambda \approx 0.13$) and there are no eigenvalues up to α = 4.000467043828088 ($b/\lambda \approx$ 0.3184). Hence, we have here a forbidden zone. If α increases further, a second set of eigenvalues appears that ends with α = 4.66533246656673. It has been found that, in an ordered medium, all eigenmodes are

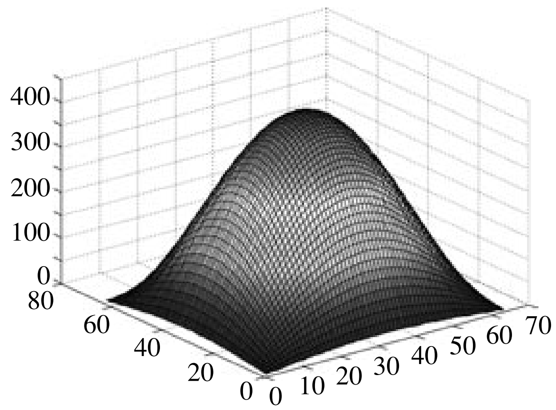

Fig. 2. Shape of the minimum mode ($b/\lambda \approx 0.0058$).

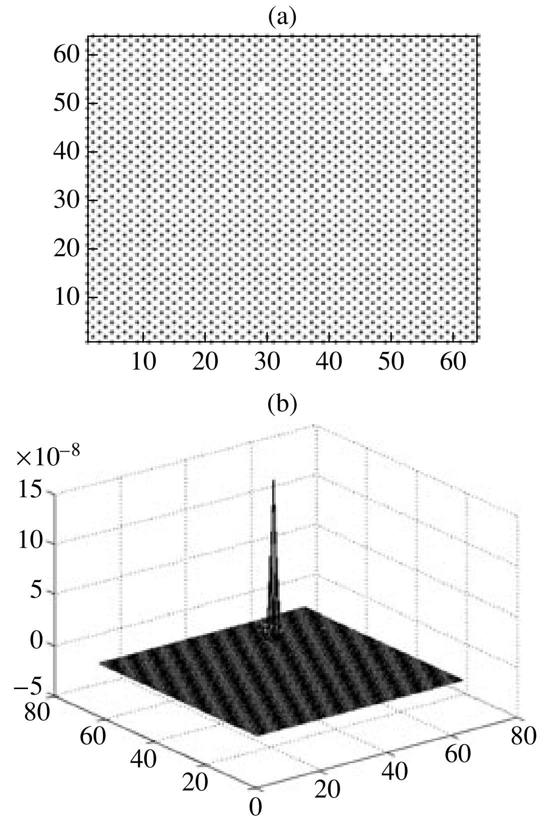

Fig. 3. (a) Ordered medium with two particles removed and (b) the shape of the mode corresponding to α = 2.21384731907047.

delocalized. An ordered medium from which several particles were removed and the order was destroyed was also considered (Fig. 3a). This medium can support localized modes whose eigenvectors lie approximately at the center of the forbidden zone (Fig. 3b).

Thus, the presence of defects in an ordered medium brings about excitation of localized resonant modes. Note that the maximum amplitude of the field of a localized mode is situated at the point occupied by the removed particle.

3. RANDOM TWO-DIMENSIONAL MEDIUM

Let us pass to the analysis of resonant modes arising in random media. We considered matrices modeling random media that are composed of $64 \times 64 = 4096$ and $128 \times 128 = 16\,384$ elements. Examples of realizations of these matrices are shown in Fig. 4.

Figures 5a and 5b present examples of localized modes obtained through calculations performed for the media modeled by the 64×64 and 128×128 matrices.

The calculations performed have shown that localized modes exist in a rather wide interval of values of ratio b/λ, approximately from 0.16 to 0.42. At $b/\lambda <$ 0.16, modes become diffuse and, at $b/\lambda \approx 0.133$, the mode is distributed over almost the entire surface 64×64 (Fig. 6a). Further decrease of ratio b/λ yields modes similar to those arising in an ordered medium (Fig. 6b), but these modes are disordered. The minimum eigenvalue is $\alpha = 0.001304456121784$ ($b/\lambda \approx 0.0057$), and the shape of the corresponding mode coincides with

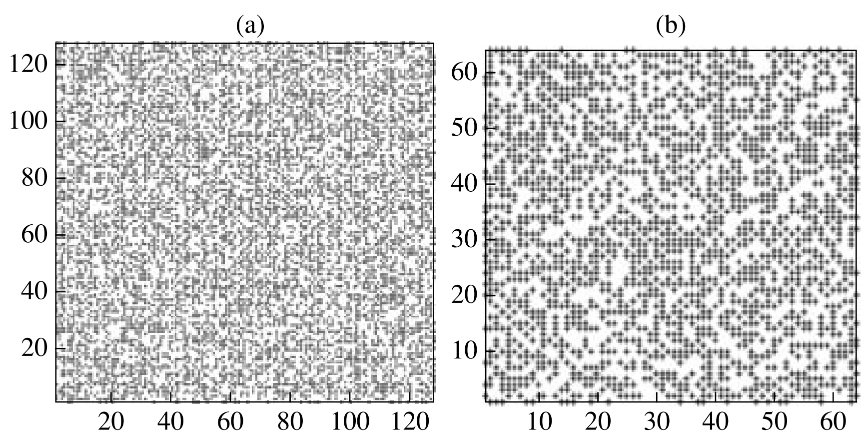

Fig. 4. Examples of realizations of the (a) 128×128 and (b) 64×64 matrices simulating random media.

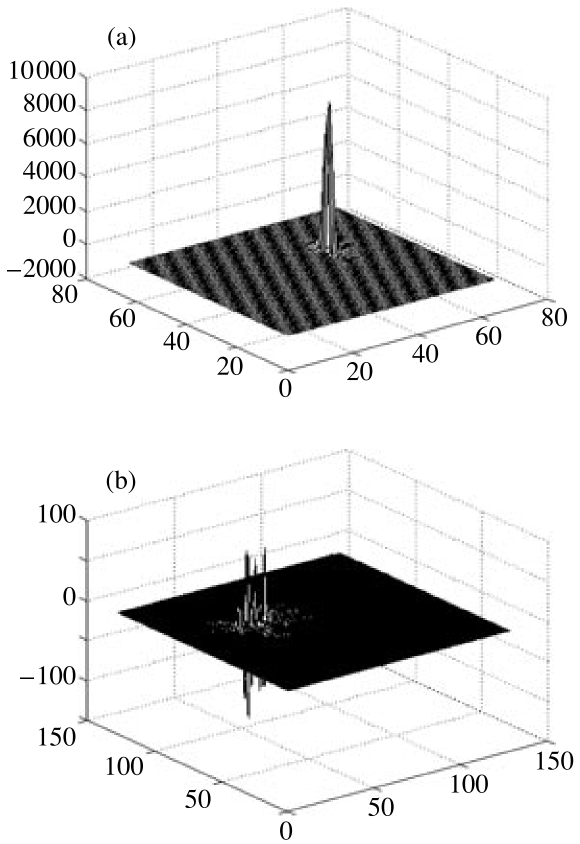

Fig. 5. Shapes of the modes corresponding to (a) α = 1.601439659937325 (b/λ ≈ 0.202) and (b) α = 3.231289318409716 (b/λ ≈ 0.286).

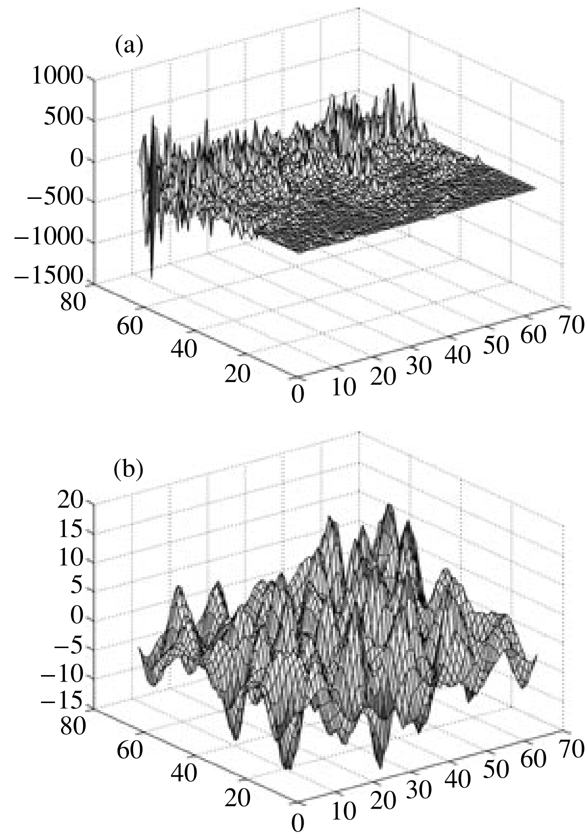

Fig. 6. (a) Diffuse mode corresponding to α = 0.7020827683999038 (b/λ ≈ 0.133) and (b) the shape of the mode corresponding to α = 0.1012976597817330 (b/λ ≈ 0.05).

that shown in Fig. 2, i.e., does not differ from that of the minimum mode in an ordered medium.

Resonant modes are localized up to the maximum eigenvalue α = 7.18356220353678 (b/λ ≈ 0.427). Note that the results presented are close to those obtained for the 128 × 128 structure and change only slightly from realization to realization.

Analysis of the distributions of localized modes shows that, as a rule, the maxima of the field amplitudes are situated in interstices between particles. The particles adjoining these interstices are also situated in the domain of existence of this mode (see Fig. 7a displaying the domain of the mode shown in Fig. 5a and the distribution of dielectric permittivity ε in this domain).

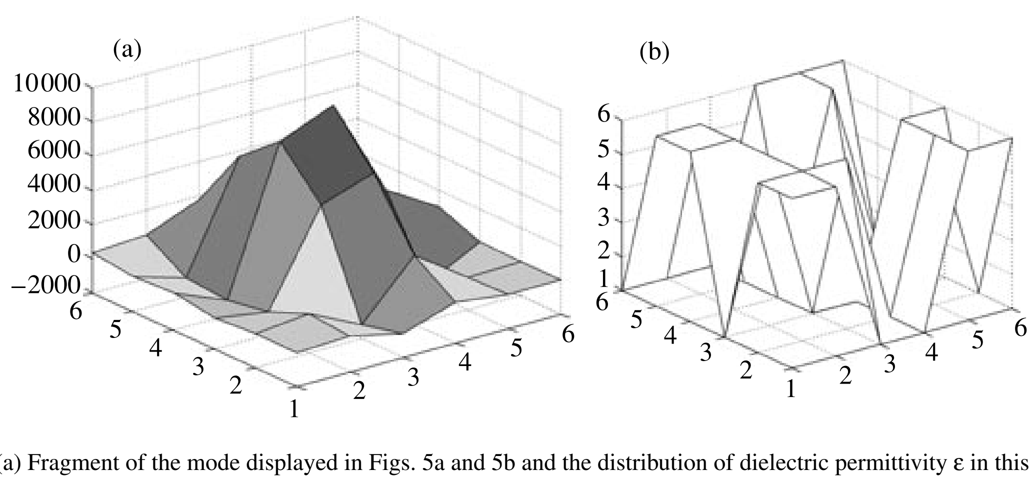

Fig. 7. (a) Fragment of the mode displayed in Figs. 5a and 5b and the distribution of dielectric permittivity ε in this domain.

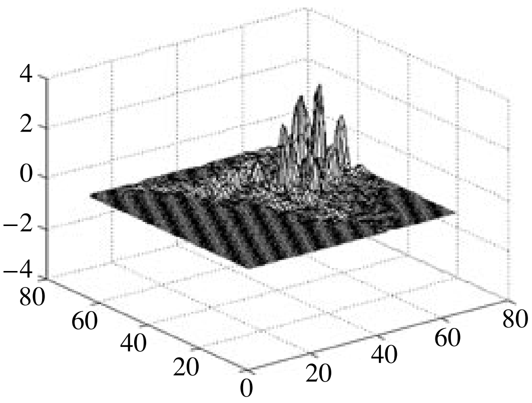

Fig. 8. Shape of the mode corresponding to $\Phi = 25\%$ and $\varepsilon = 6$ ($\alpha = 1.602833074446612$).

In order to analyze the role of the scattering intensity, we estimated the effect of the filling factor on the mode type. It has been found that modes become delocalized (for $\varepsilon = 6$) at $\Phi < 40\%$. Figure 8 presents the shape of the mode corresponding to $\Phi = 25\%$.

In order to estimate the role of the dielectric permittivity of a medium, we considered a medium with $\varepsilon = 3$ ($\Phi = 50\%$). It has been found that the lower boundary of the interval of values of ratio b/λ, that is, the lower boundary of the interval in which resonant modes are localized, is approximately 0.32; i.e., this boundary is approximately twice as high as that corresponding to $\varepsilon = 6$. The upper limit for values of ratio b/λ at which resonant modes can exist is almost the same as in the case of $\varepsilon = 6$; the modes are localized up to the maximum eigenvalue. As an example, Fig. 9a shows the field of the localized mode corresponding to $\varepsilon = 3$. In this case, the filling factor plays an important role: the modes become completely delocalized even at $\Phi = 25\%$ (Fig. 9b).

Hence, we have found that resonant modes excited in a random 2D medium exist in a limited interval of values of ratio b/λ. The upper boundary of this interval (~0.42) is almost independent of ε, whereas the lower boundary slightly increases as ε decreases. The upper

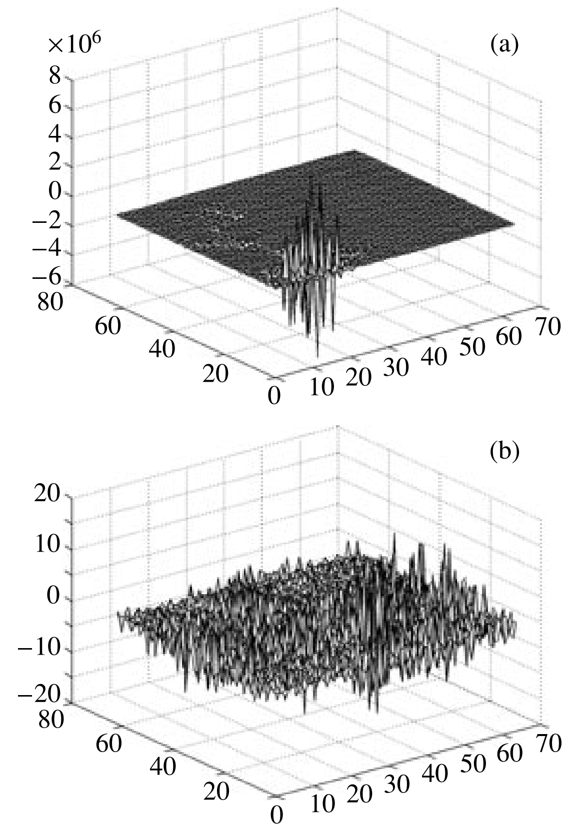

Fig. 9. Shapes of (a) the localized mode existing at $\varepsilon = 3$, $\Phi = 50\%$, and $b/\lambda \approx 0.374$ and (b) the delocalized mode existing at $\varepsilon = 3$, $\Phi = 25\%$, and $b/\lambda \approx 0.356$.

part of this interval corresponds to localized modes and the lower part corresponds to delocalized modes. The boundary between these parts corresponds to the value of ratio b/λ that is almost inversely proportional to ε.

4. ONE-DIMENSIONAL AND THREE-DIMENSIONAL MEDIA

The proposed simplified method can be used for the analysis of media of an arbitrary dimension. In any case, this method is reduced to solution of a system of equations similar to system (1). The amplitude of field

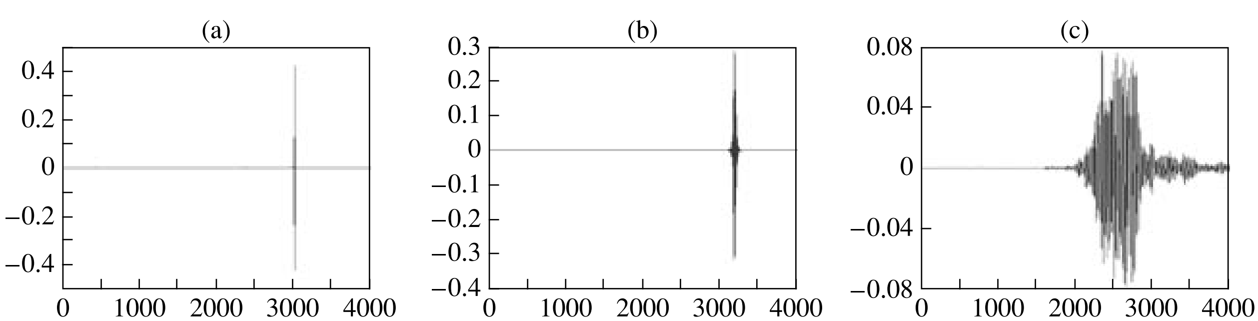

Fig. 10. Shapes of the modes corresponding to (a) the maximum α, (b) $\alpha = 0.2013685622694161$, and (c) $\alpha = 0.01979756426258023$.

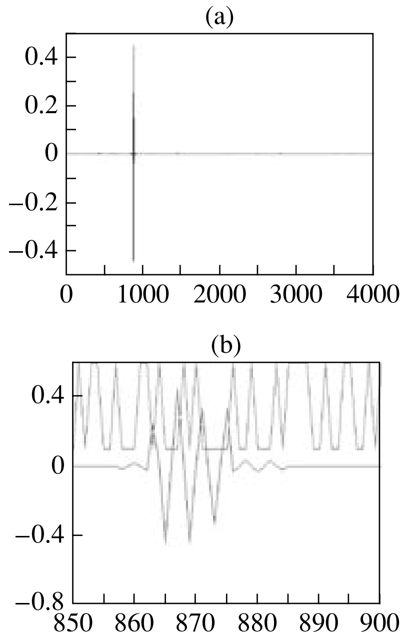

Fig. 11. (a) Shape of the mode corresponding to α = 2.022305428283262 (b) and the distribution of dielectric permittivity ε in this domain.

E in a 1D medium should have one index (i), while the field amplitude in a 3D medium should have three indices (i, j, k); cipher 4 in parentheses should be replaced with 2 for a 1D medium and 6 for a 3D medium. Moreover, in the case of a 1D medium, terms with indices $j + 1$ and $j - 1$ disappear and, in the case of a 3D medium, the system contains additional terms with indices $k + 1$ and $k - 1$.

A. One-Dimensional Medium

A 1D medium was modeled by a matrix containing one row (a vector) whose elements take on two values. As in the case of a 2D medium, one of these values was equal to the dielectric permittivity of the material particles ($\varepsilon = 6$) and the other value was equal to the dielectric permittivity of interstices between particles ($\varepsilon_1 = 1$). In numerical calculations, the length of this vector was equal to 4000 and the filling factor was $\Phi = 50\%$.

For a random 1D medium, we obtained resonant modes in the interval from $\alpha = 1.782043700533113 \times 10^{-7}$ ($b/\lambda \approx 0.672 \times 10^{-4}$) to $\alpha = 3.920375036418377$ ($b/\lambda \approx 0.315$). From approximately $\alpha = 0.2$ to the maximum value of eigenvalue α, modes are localized (Figs. 10a, 10b) and, at $\alpha < 0.2$, modes are delocalized (Fig. 10c). This result varies only slightly from realization to realization.

In order to find positions of the field maxima for localized modes, we considered the mode corresponding to $\alpha = 2.022305428283262$ (Fig. 11). It can be seen readily that positions of the field maxima correspond to $\varepsilon = 1$, i.e., to interstices between particles.

Thus, as in the case of the 2D medium, the cavity is seemingly formed by an interstice and surrounding particles. For the sake of completeness, we considered also the following ordered variant of the same 1D medium (4000 elements): only even elements of the vector-medium were equal to 6. It has been found that, in this case, resonant modes exist between $\alpha_1 = 1.761548244307406 \times 10^{-7}$ and $\alpha_2 = 0.3333332716791421$ and between $\alpha_3 = 2.000000061654191$ and $\alpha_4 = 2.33333315717851$. There are no resonant modes between α_2 and α_3; this interval is a forbidden zone. If we remove some particles from the medium under study (break the order), localized modes appear in the forbidden zone. For example, removal of three particles (elements with numbers equal to 1000, 1050, and 1100) causes appearance of three localized modes whose eigenvalues differ from each other in only the 16th decimal digit (Fig. 12).

In order to determine positions of the maxima of amplitudes of these modes, Fig. 13 shows corresponding field distributions near the locations from which the particles were removed. It is seen that the maxima are situated at the points corresponding to positions of removed particles. Thus, maxima of amplitudes of the

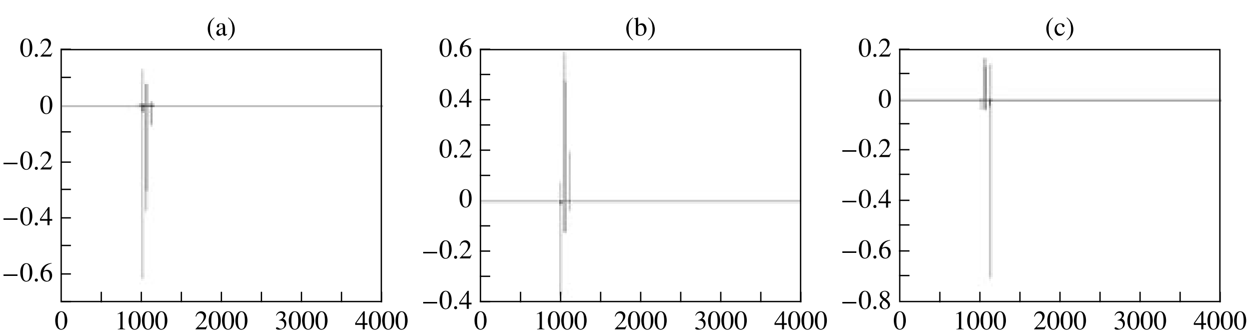

Fig. 12. Modes arising in the forbidden zone at (a) α = 0.7394482956983173, (b) α = 0.7394482956983177, and (c) α = 0.7394482956983179.

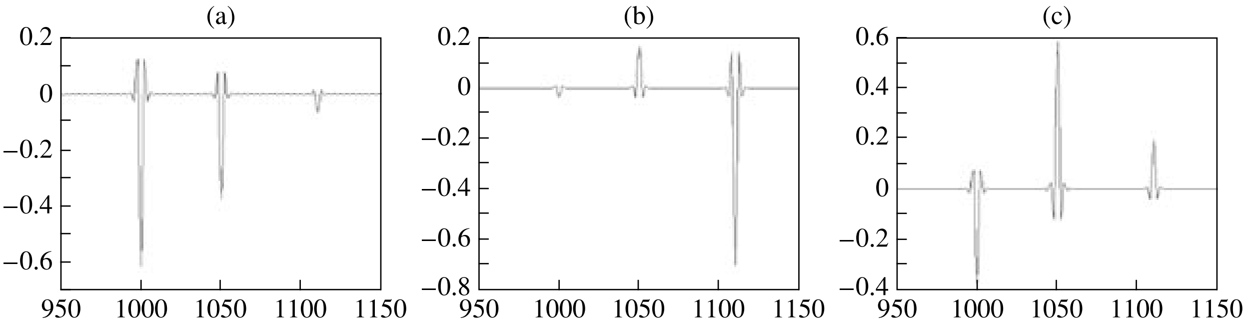

Fig. 13. Modes shown in Fig. 12 near the location with particles removed.

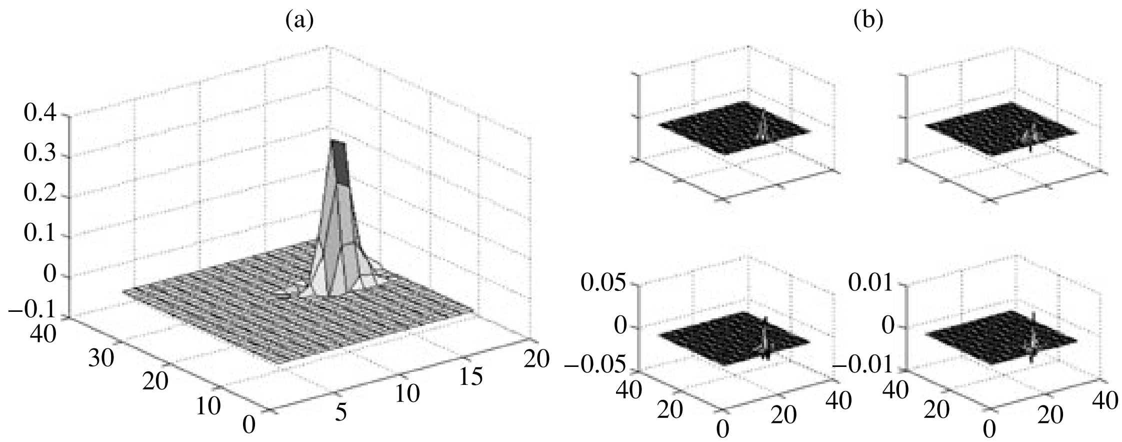

Fig. 14. Localized mode corresponding to α = 2.304185695194057: (a) the *YZ* section at *X* = 6 and (b) the *XY* sections at *Z* = 13–16.

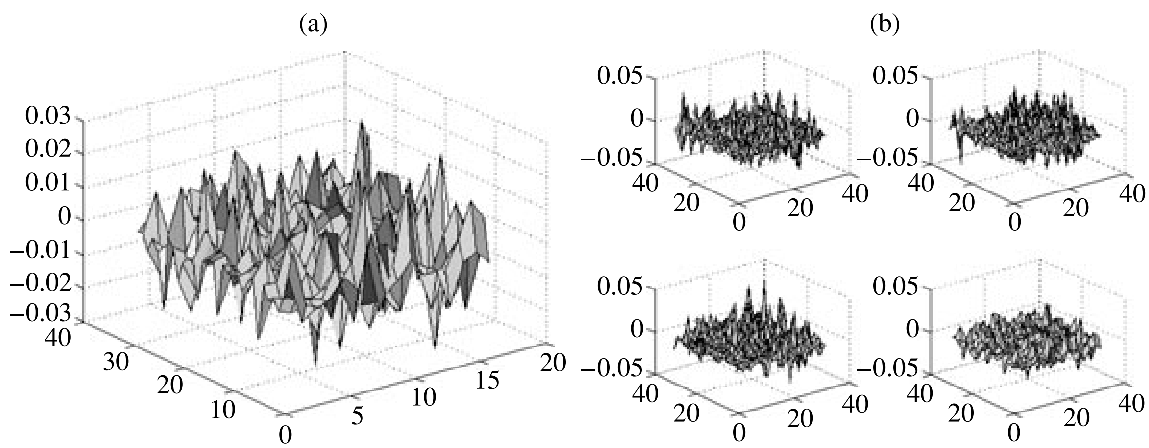

Fig. 15. Nonlocalized mode corresponding to α = 0.9998284686596601: (a) the *YZ* section at *X* = 28 and (b) the *XY* sections at *Z* = 13–16.

fields of localized modes are again situated in interstices between the particles forming the medium.

B. A Three-Dimensional Medium

A 3D medium was modeled by a set of matrix-layers placed in common coordinate system (*X*, *Y*, *Z*), where the layer number determined the third coordinate (*Z*). Elements of these matrices randomly took on values corresponding to the dielectric permittivity of the material particles ($\varepsilon = 6$) and the dielectric permittivity of interstices between the particles ($\varepsilon_1 = 1$). An individual random set of values of dielectric permittivity ε was used in each layer. At this point we had considered

Table

ε	α_{min} (b/λ)	α_{max} (b/λ)	α_b (b/λ)	α_t (b/λ)	Δ
6	0.01472 (0.0193)	6.98457 (0.421)	1.00174 (0.159)	5.99664 (0.39)	4.995
3	0.0258 (0.0256)	7.97346 (0.45)	2.00395 (0.225)	5.99387 (0.39)	3.99

Note: Here, α_{min} and α_{max} are the minimum and the maximum eigenvalue, respectively; α_t and α_b are the eigenvalues corresponding, respectively, to the top and bottom of the forbidden zone; ciphers in parentheses are values of ratio b/λ; and Δ is the width of the forbidden zone.

16 layers, each representing a 32 × 32 matrix with the filling factor $\Phi = 50\%$. Amplitude distributions of the eigenmode fields in the 3D medium are shown as sections of these distributions in the *XY* plane at several values of coordinate *Z* and sections in the *YZ* plane at a fixed value of coordinate *X*. In a random 3D medium both localized (Fig. 14) and delocalized (Fig. 15) eigenmodes exist.

It has been found that, for the considered 3D medium with $\varepsilon = 6$ and $\Phi = 50\%$, the minimum and maximum eigenvalues are $\alpha = 0.01408380580808201$ ($b/\lambda \approx 0.0189$) and $\alpha = 10.30571606048049$ ($b/\lambda \approx 0.511$); localized modes exist at $b/\lambda > 0.195$ ($\alpha > 1.500592624112003$). These results change only slightly from realization to realization.

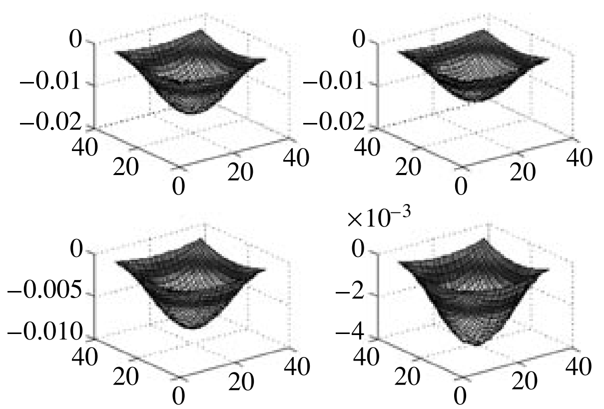

Fig. 16. Shape of the mode corresponding to α_{min} = 0.01472336304419967 (the *XY* sections at *Z* = 13–16).

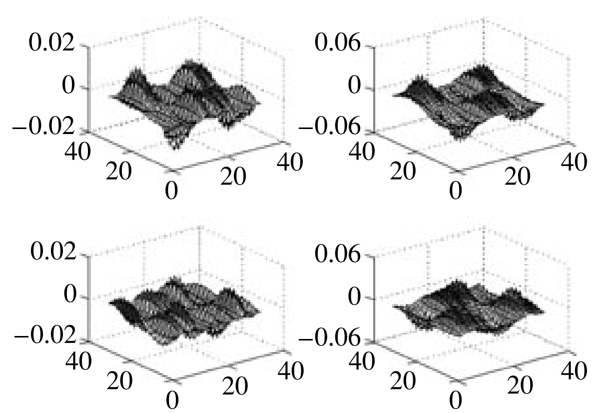

Fig. 17. Shape of the mode corresponding to α = 0.03397834896789619 (the *XY* sections at *Z* = 13–16).

An ordered 3D medium was modeled with matrix-layers in which elements equal to 6 are arranged in staggered order. The layers were placed so that this order was retained in the vertical section as well. Eigenmodes existing in this medium are nonlocalized. Examples of the *XY* sections of these modes are shown in Figs. 16 and 17.

Ordered media containing staggered elements with $\varepsilon = 6$ and 3 were compared to each other. The results obtained are listed in the table. It is of interest that eigenvalue α_b corresponding to $\varepsilon = 3$ is almost twice as large as the same eigenvalue corresponding to $\varepsilon = 6$. Hence, values of ratio b/λ corresponding to the bottom of the forbidden zone are inversely proportional to the refractive index ($\varepsilon = n^2$).

CONCLUSIONS

A simplified method for modeling eigenmodes excited in inhomogeneous media has been proposed. The method is based of the numerical solution of the steady-state wave equation with a specified distribution of dielectric permittivity. By means of discretization, the wave equation is reduced to a system of homogeneous linear equations for which it is possible to determine eigenvalues $\alpha = (2\pi b/\lambda)^2$, where b is the discretization step and λ is the radiation wavelength. To each eigenvalue α an eigenvector specifying the spatial distribution of the field amplitude, i.e., the mode shape, corresponds.

This method was used for the analysis of random and ordered media of different dimensions. The media were modeled by matrices whose elements took on two values. One of these values corresponded to the dielectric permittivity of the material particles and the other value corresponded to the dielectric permittivity of interstices between particles.

Numerical calculations performed with the use of this method have shown that localized resonant modes arise only in the absence of complete ordering and that effective cavities forming such modes comprise interstices between particles and adjoining particles. Maxima of the field amplitude are situated in interstices between particles. It has also been found that, in random media, resonant modes exist in a fixed interval of values of the ratio of the particle size and the wavelength ($b/\lambda < 1$). Moreover, localized modes arise at larger values of this ratio than those for delocalized modes. It has been found that the value of ratio b/λ corresponding to the boundary between delocalized and localized modes is almost inversely proportional to dielectric permittivity ε of the particle material.

The use of the proposed method for the analysis of ordered inhomogeneous media simulating photonic crystals allows a forbidden zone to be revealed. In the 3D variant of an ordered medium with $\varepsilon = 3$, eigenvalue α_b, determining the position of the bottom of the forbidden zone, was found to be approximately twice as large as the same eigenvalue in the medium with $\varepsilon = 6$; i.e., values of ratio b/λ corresponding to the bottom of the forbidden zone are inversely proportional to the refractive index. Hence, the position of the long-wave edge of the forbidden zone shifts proportionally with the dielectric permittivity of the material of the particles forming the medium.

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